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Both compact and sequentially compact sets in abelian topological group

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ABSTRACT

We show that every abelian topological group contains many interesting sets which are both compact and sequentially compact. Then we can deduce some useful facts, e.g.,

(1) if G is a Hausdorff abelian topological group and $\mu : 2^{\mathbb{N}} \to G$ is countably additive, then the range $\mu(2^{\mathbb{N}}) = \{\mu(A)\colon A \subseteq \mathbb{N}\}$ is compact metrizable;
(2) if X is a Hausdorff locally convex space and $\{x_j\} \subset X$, then $F = \{\sum_{j\in\Delta} x_j\colon \Delta \subset \mathbb{N}, \Delta \text{ is finite}\}$ is relatively compact in $(X, \textit{weak})$ if and only if F is relatively compact in X, and if and only if F is relatively compact in $(X, \mathscr{F}(\mathfrak{M}))$ where $\mathscr{F}(\mathfrak{M})$ is the Dierolf topology which is the strongest $\langle X, X'\rangle$-polar topology having the same subseries convergent series as the weak topology.

We would like to say that the following Theorem A is a typical example of basic facts which are very simple but very interesting [1, p. 39].

Theorem A. *([2, Theorem 1], [3, Corollary 10]) Let G be an abelian topological group and Ω an arbitrary nonempty set. For each $j \in \mathbb{N}$, $f_j : \Omega \to G$ is a mapping such that there is an $\omega_0 \in \Omega$ such that $f_j(\omega_0) = 0$ for all j. Then the following (α) and (β) are equivalent:*

(α) *$\sum_{j=1}^{\infty} f_j(\omega_j)$ converges for every $\{\omega_j\} \subset \Omega$;*
(β) *$\sum_{j=1}^{\infty} f_j(\omega_j)$ converges uniformly for $\{\omega_j\} \subset \Omega$.*

Seeing the Tychonoff's product theorem, the following theorem is a direct consequence of Theorem A.

Theorem B. *([2, Corollary 3]) Let X be a topological vector space and $\{x_j\}$ a sequence in X such that $\sum_{j=1}^{\infty} t_j x_j$ converges for each $\{t_j\} \in l^\infty = \{\{s_j\} \subset \mathbb{C}\colon \sup_j |s_j| < +\infty\}$. Then the set $\{\sum_{j=1}^{\infty} t_j x_j\colon \sup_j |t_j| \leqslant 1\}$ is both compact and sequentially compact and hence, letting $T(\{t_j\}) = \sum_{j=1}^{\infty} t_j x_j$ for $\{t_j\} \in l^\infty$, $T : l^\infty \to X$ is both a compact and sequentially compact operator, i.e., for every bounded $B \subset X$, $\overline{T(B)}$ is both compact and sequentially compact.*

With the aid of Theorem B, [2] has obtained a very nice result as follows:

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Theorem C. *([2, Theorem 4]) For a sequentially complete locally convex space X, the following* (a), (b) *and* (c) *are equivalent*:

(a) *X contains no copy of c_0, i.e., X has no subspace which is linearly homeomorphic to $(c_0, \|\cdot\|_\infty)$;*
(b) (*Bessaga–Pelczynsky*) *Every weakly unconditionally Cauchy series on X is unconditionally convergent*;
(c) *Every continuous linear operator $T: c_0 \to X$ is both compact and sequentially compact.*

It is also worthwhile observing that Theorem A has played a key role in the proof of the following very nice result:

Theorem D. *([3, Theorem 7], [4, Theorem B]) Let X, Y be topological vector spaces and $f_{ij}: X \to Y$ a mapping such that $f_{ij}(0) = 0$ for all $i, j \in \mathbb{N}$. If the matrix $(f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), c(Y))$, i.e., $\lim_i \sum_{j=1}^\infty f_{ij}(x_j)$ exists for each bounded $\{x_j\} \subset X$, then for every bounded $B \subset X$ the series $\sum_{j=1}^\infty f_{ij}(x_j)$ converges uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$, and $\lim_i f_{ij}(x)$ exists whenever $j \in \mathbb{N}$ and $x \in X$. If Y is sequentially complete, then the converse also holds.*

In this paper, we would like to develop Theorem B and then give a series of interesting facts in topology and analysis. Throughout this note, abelian topological groups are Hausdorff, and $\mathscr{N}(G)$ is the family of neighborhoods of 0 in the abelian topological group G.

First, a very simple argument which is similar to the proof of Theorem A [2, p. 206] gives an improvement of Theorem A as follows.

Lemma 1. *Let G be an abelian topological group and Ω an arbitrary nonempty set. For a mapping sequence $\{f_j\} \subset G^\Omega$, the following* (α) *and* (β) *are equivalent*:

(α) *$\sum_{j=1}^\infty f_j(\omega_j)$ converges for every $\{\omega_j\} \subset \Omega$;*
(β) *$\sum_{j=1}^\infty f_j(\omega_j)$ converges uniformly for $\{\omega_j\} \subset \Omega$.*

Then we have the following basic proposition.

Theorem 1. *Let Ω be a compact* (*resp., sequentially compact*) *space and G an abelian topological group. If $\{f_j\} \subset C(\Omega, G)$ is such that $\sum_{j=1}^\infty f_j(\omega_j)$ converges for each $\{\omega_j\} \subset \Omega$, then $\{\sum_{j=1}^\infty f_j(\omega_j)\colon \omega_j \in \Omega,\ \forall j \in \mathbb{N}\}$ is compact* (*resp., sequentially compact*).

Proof. Define $F: \Omega^{\mathbb{N}} \to G$ by $F(\{\omega_j\}) = \sum_{j=1}^\infty f_j(\omega_j)$. As in the proof of [2, Cor. 2], Lemma 1 implies that F is continuous. Then the desired conclusion follows from Tychonoff's product theorem and the diagonal procedure. □

Even some simple special cases of Theorem 1 can have interesting applications, e.g., $\Omega = \{t \in \mathbb{C}\colon |t| \leqslant 1\}$ is both compact and sequentially compact and so we have

Corollary 1 *(Theorem B). Let X be a topological vector space and $\{x_j\} \subset X$. If $\sum_{j=1}^\infty t_j x_j$ converges for each bounded $\{t_j\} \subset \mathbb{C}$, then $\{\sum_{j=1}^\infty t_j x_j\colon \sup_j |t_j| \leqslant 1\}$ is both compact and sequentially compact.*

Let G be an abelian topological group, $\{x_j\} \subset G$ and $\Delta \subseteq \mathbb{N}$. For $\Delta = \emptyset$, let $\sum_{j\in\Delta} x_j = 0$ and for $\Delta = \{j_1, j_2, \ldots\}$ with $j_1 < j_2 < j_3 < \cdots$, let $\sum_{j\in\Delta} x_j = \sum_{k=1}^\infty x_{j_k}$.

Corollary 2. *Let G be an abelian topological group and $\{x_j\} \subset G$. If $\sum_j x_j$ is subseries convergent, i.e., $\sum_{j\in\Delta} x_j$ converges for each $\Delta \subseteq \mathbb{N}$, then the set $\{\sum_{j\in\Delta} x_j\colon \Delta \subseteq \mathbb{N}\}$ is both compact and sequentially compact.*

Proof. As a subspace of $\mathbb{R}$, $\Omega = \{0, 1\}$ is both compact and sequentially compact. For each $j \in \mathbb{N}$, define $f_j: \Omega \to G$ by $f_j(0) = 0$ and $f_j(1) = x_j$. Then each f_j is continuous.

For $\{\omega_j\} \subset \Omega$ and $\Delta = \{j \in \mathbb{N}\colon \omega_j = 1\}$, we have $\sum_{j=1}^\infty f_j(\omega_j) = \sum_{j\in\Delta} f_j(1) = \sum_{j\in\Delta} x_j \in G$. Then the desired conclusion follows from Theorem 1. □

For the range of a vector measure, there is a nice Uhl's theorem saying that if X is a Banach space with the Radon–Nykodym property and Σ is a σ-algebra of subsets of a set S and $\mu: \Sigma \to X$ is of bounded variation, nonatomic and countably additive, then $\overline{\mu(\Sigma)} = \overline{\{\mu(A)\colon A \in \Sigma\}}$ is compact in X ([5], [6, p. 266]).

It is interesting that a very strong result holds when the σ-algebra is $2^{\mathbb{N}} = \{A\colon A \subseteq \mathbb{N}\}$.

Theorem 2. *Let G be an abelian topological group. Then for every countably additive $\mu: 2^{\mathbb{N}} \to G$, the range $\mu(2^{\mathbb{N}}) = \{\mu(A)\colon A \subseteq \mathbb{N}\}$ is both compact and sequentially compact. Moreover, if Σ is a σ-algebra of subsets of a set Ω and $\mu: \Sigma \to G$ is countably additive, then for every pairwise disjoint $\{A_j\} \subset \Sigma$, the set $\{\sum_{j\in\Delta} \mu(A_j)\colon \Delta \subseteq \mathbb{N}\}$ is both compact and sequentially compact.*

Proof. If $i, j \in \mathbb{N}$ and $i \neq j$, then $\{i\} \cap \{j\} = \emptyset$. For every $A \subseteq \mathbb{N}$, $\sum_{j\in A} \mu(\{j\}) = \mu(\bigcup_{j\in A}\{j\}) = \mu(A)$ and so $\sum_j \mu(\{j\})$ is subseries convergent. Then the desired conclusion follows from Corollary 2.

If Σ is a σ-algebra and $\{A_j\}$ is pairwise disjoint in Σ and $\mu : \Sigma \to G$ is countably additive, then $\sum_j \mu(A_j)$ is subseries convergent. Then $\{\sum_{j\in\Delta} \mu(A_j)\colon\ \Delta \subseteq \mathbb{N}\}$ is both compact and sequentially compact by Corollary 2. $\square$

Observe that in the above applications of Theorem 1, the compact space Ω is metrizable: $\Omega = \{0, 1\}$ or $\{t \in \mathbb{C}\colon\ |t| \leqslant 1\}$.

Theorem 3. *Let Ω be a compact metrizable space and G a Hausdorff abelian topological group. If $\{f_j\} \subset C(\Omega, G)$ is such that $\sum_{j=1}^{\infty} f_j(\omega_j)$ converges for each $\{\omega_j\} \subset \Omega$, then $\{\sum_{j=1}^{\infty} f_j(\omega_j)\colon\ \omega_j \in \Omega$ for all $j\}$ is compact and metrizable.*

Proof. Say that $\Omega = (\Omega, d)$ is compact and pseudometric. Then $\Omega^{\mathbb{N}} = (\Omega^{\mathbb{N}}, \rho)$ is also compact and pseudometric, where

$$\rho\big(\{\omega_j\}, \{\lambda_j\}\big) = \sum_{j=1}^{\infty} 2^{-j} \frac{d(\omega_j, \lambda_j)}{1 + d(\omega_j, \lambda_j)}.$$

Define $F : \Omega^{\mathbb{N}} \to G$ by $F(\{\omega_j\}) = \sum_{j=1}^{\infty} f_j(\omega_j)$. As in the proof of Theorem 1, F is continuous. Since G is Hausdorff, $F(\Omega^{\mathbb{N}})$ is a compact metrizable subspace of G [7, Corollary 23.2, p. 166]. $\square$

Corollary 3.

(a) *If X is a Hausdorff topological vector space and $\{x_j\}$ is a sequence in X such that $\sum_{j=1}^{\infty} t_j x_j$ converges for each bounded $\{t_j\} \subset \mathbb{C}$, then $\{\sum_{j=1}^{\infty} t_j x_j\colon\ \sup_j |t_j| \leqslant 1\}$ is compact and metrizable.*
(b) *If G is a Hausdorff abelian topological group and $\{x_j\}$ is a sequence in G such that $\sum x_j$ is subseries convergent, then $\{\sum_{j\in\Delta} x_j\colon\ \Delta \subseteq \mathbb{N}\}$ is compact and metrizable.*
(c) *If G is a Hausdorff abelian topological group and $\mu : 2^{\mathbb{N}} \to G$ is countably additive, then the range $\mu(2^{\mathbb{N}}) = \{\mu(A)\colon\ A \subseteq \mathbb{N}\}$ is compact and metrizable.*

Proof. We prove (c) only. For $A = \{n_1, n_2, \ldots\} \subseteq \mathbb{N}$ and every rearrangement $\{m_1, m_2, \ldots\}$ of A, $\sum_{k=1}^{\infty} \mu(\{n_k\}) = \mu(\bigcup_{k=1}^{\infty}\{n_k\}) = \mu(A) = \mu(\bigcup_{k=1}^{\infty}\{m_k\}) = \sum_{k=1}^{\infty} \mu(\{m_k\})$. Let $\Omega = \{0, 1\}$ and $f_j(0) = 0$, $f_j(1) = \mu(\{j\})$ for $j \in \mathbb{N}$. Then each $f_j : \Omega \to G$ is continuous. Define $F : \Omega^{\mathbb{N}} \to G$ by $F(\{\omega_j\}) = \sum_{j=1}^{\infty} f_j(\omega_j) = \sum_{\omega_j=1} f_j(1) = \sum_{\omega_j=1} \mu(\{j\}) = \mu(\{j\colon\ \omega_j = 1\})$. Then $\mu(2^{\mathbb{N}}) = \{\mu(A)\colon\ A \subseteq \mathbb{N}\} = \{\sum_{j\in A} \mu(\{j\})\colon\ A \subseteq \mathbb{N}\} = \{\sum_{j\in A} f_j(1)\colon\ A \subseteq \mathbb{N}\} = \{\sum_{j\in A} f_j(1) + \sum_{j\in\mathbb{N}\setminus A} f_j(0)\colon\ A \subseteq \mathbb{N}\} = \{F(\chi_A)\colon\ A \subseteq \mathbb{N}\} = F(\Omega^{\mathbb{N}})$. By Theorem 3, $\mu(2^{\mathbb{N}})$ is compact metrizable. $\square$

Theorem 4. *Let X be a Hausdorff topological vector space, $\{x_j\} \subset X$ and $F = \{\sum_{j\in\Delta} x_j\colon\ \Delta \subset \mathbb{N},\ \Delta$ is finite$\}$. The following* (1), (2), (3) *and* (4) *are equivalent*:

(1) *$\sum_j x_j$ is subseries convergent*;
(2) *$\overline{F}$ is compact*;
(3) *$\overline{F}$ is sequentially compact*;
(4) *$\overline{F}$ is both compact and sequentially compact.*

Proof. (1) $\Rightarrow$ (4). By Corollary 2, (1) implies that $S = \{\sum_{j\in\Delta} x_j\colon\ \Delta \subseteq \mathbb{N}\}$ is both compact and sequentially compact. Since X is Hausdorff, S is closed and so $\overline{F} \subseteq \overline{S} = S \subseteq \overline{F}$, i.e., $\overline{F} = S$. Thus, (1) implies (4).

(2) $\Rightarrow$ (1). Let $j_1 < j_2 < \cdots$ in $\mathbb{N}$. If $\{\sum_{k=1}^{n} x_{j_k}\}_{n=1}^{\infty}$ is not Cauchy, then there exist $U \in \mathscr{N}(X)$ and integers $m_1 \leqslant n_1 < m_2 \leqslant n_2 < \cdots$ such that $y_p = \sum_{k=m_p}^{n_p} x_{j_k} \notin U$, $\forall p \in \mathbb{N}$. Pick a balanced $V \in \mathscr{N}(X)$ for which $V + V \subset U$. Since $\overline{F}$ is compact, F is bounded and so $F \subset lV$ for some $l \in \mathbb{N}$. Then pick a balanced $W \in \mathscr{N}(X)$ such that $\overbrace{W + \cdots + W}^{l\,\text{times}} \subset V$. Since $\{y_1, y_2, \ldots\} \subset F \subset \overline{F}$ and $\overline{F}$ is compact, $\{y_1, y_2, \ldots\}$ is totally bounded [8, p. 83] and so there is an $n_0 \in \mathbb{N}$ such that

$$\{y_1, y_2, \ldots\} = \{y_p\}_{p=1}^{\infty} \subset \bigcup_{p=1}^{n_0} (y_p + W)$$

and so there is a $p_0 \in \{1, 2, \ldots, n_0\}$ for which $y_{p_0} + W$ contains infinitely many elements of $\{y_p\}_{p=1}^{\infty}$, say $\{y_{p_i}\}_{i=1}^{\infty} \subset y_{p_0} + W$. Then

$$\sum_{i=1}^{l} y_{p_i} \in \overbrace{(y_{p_0} + W) + (y_{p_0} + W) + \cdots + (y_{p_0} + W)}^{l\,\text{times}} = l y_{p_0} + \overbrace{W + \cdots + W}^{l\,\text{times}} \subset l y_{p_0} + V,$$

that is,

$$\sum_{i=1}^{l} y_{p_i} = l y_{p_0} + v \quad \text{for some } v \in V$$

and so

$$l y_{p_0} \in \sum_{i=1}^{l} y_{p_i} - V = \sum_{i=1}^{l} y_{p_i} + V = \sum_{i=1}^{l} \sum_{k=m_{p_i}}^{n_{p_i}} x_{j_k} + V \subset F + V \subset lV + V \subset lU$$

since if $x, y \in V$, then $lx + y = l(x + \frac{y}{l}) \in l(V + V) \subset lU$. Then $y_{p_0} \in U$, a contradiction. Thus, $\{\sum_{k=1}^{n} x_{j_k}\}_{n=1}^{\infty}$ is a Cauchy sequence in the compact $\overline{F}$. Since compact $\overline{F}$ is complete [8, p. 88], $\sum_{k=1}^{\infty} x_{j_k} = \lim_n \sum_{k=1}^{n} x_{j_k}$ exists.

(3) $\Rightarrow$ (1). Since $\overline{F}$ is sequentially compact, $\overline{F}$ is countably compact [9, p. 125] and so $\overline{F}$ is totally bounded [8, p. 86]. Hence, as in the proof of (2) $\Rightarrow$ (1), for every $j_1 < j_2 < \cdots$ in $\mathbb{N}$, $\{\sum_{k=1}^{n} x_{j_k}\}_{n=1}^{\infty}$ is Cauchy in the sequentially compact $\overline{F}$ and so there is a sequence of integers $n_1 < n_2 < \cdots$ such that $\sum_{k=1}^{\infty} x_{j_k} = \lim_i \sum_{k=1}^{n_i} x_{j_k} \in \overline{F}$. $\square$

Note that Robertson has established a series of basic results connecting convergence of series and compactness of finite partial sums in [10] and [11].

Theorem 5. *Let X be a Hausdorff topological vector space, $\{x_j\} \subset X$ and $F = \{\sum_{j\in\Delta} t_j x_j\colon \Delta \text{ is finite},\ \sup_j |t_j| \leqslant 1\}$. The following* (i), (ii), (iii) *and* (iv) *are equivalent*:

(i) $\sum_j t_j x_j$ *converges for each* $(t_j) \in l^\infty$;
(ii) $\overline{F}$ *is compact*;
(iii) $\overline{F}$ *is sequentially compact*;
(iv) $\overline{F}$ *is both compact and sequentially compact.*

Proof. (i) $\Rightarrow$ (iv). By Corollary 1 (= Theorem B), (i) implies that $S = \{\sum_{j=1}^{\infty} t_j x_j\colon \sup_j |t_j| \leqslant 1\}$ is both compact and sequentially compact and so $S = \overline{S}$ in the Hausdorff space X. Thus $\overline{F} \subset \overline{S} = S \subset \overline{F}$ and (i) $\Rightarrow$ (iv).

(ii) $\Rightarrow$ (i). Let $(s_j) \in l^\infty$ with $0 < \sup_j |s_j| < +\infty$. By (ii), $\overline{\{\sum_{j\in\Delta} \frac{s_j}{\sup_j |s_j|} x_j\colon \Delta \text{ is finite}\}}$ is compact. By Theorem 4, $\sum_j s_j x_j = \sup_j |s_j| \sum_j \frac{s_j}{\sup_j |s_j|} x_j$ is subseries convergent and, in particular, $\sum_{j=1}^{\infty} s_j x_j$ converges.

Similarly, (iii) $\Rightarrow$ (i) holds. $\square$

For a locally convex space X and its dual X', let $\mathscr{F}(\mathfrak{M})$ be the Dierolf topology on X. $\mathscr{F}(\mathfrak{M})$ is the strongest $\langle X, X'\rangle$-polar topology on X which has the same subseries convergent series as the weak topology $\sigma(X, X')$ [12, Theorem 2.2]. In general,

$$\begin{aligned} \text{the weak topology } \sigma(X, X') &\subseteq \text{the locally convex topology on } X \\ &\subseteq \text{the Mackey topology } \tau(X, X') \\ &\subseteq \text{the Dierolf topology } \mathscr{F}(\mathfrak{M}). \end{aligned}$$

We have the following invariant result.

Theorem 6. *Let X be a Hausdorff topological vector space, $\{x_j\} \subset X$ and $F = \{\sum_{j\in\Delta} t_j x_j\colon \Delta \text{ is finite},\ \sup_j |t_j| \leqslant 1\}$. Then we have the following*:

(I) *F is relatively compact in $(X, weak)$ if and only if F is relatively compact in $(X, \mathscr{F}(\mathfrak{M}))$*;
(II) *F is relatively sequentially compact in $(X, weak)$ if and only if F is relatively sequentially compact in $(X, \mathscr{F}(\mathfrak{M}))$*;
(III) *F is both relatively compact and relatively sequentially compact in $(X, weak)$ if and only if F is both relatively compact and relatively sequentially compact in $(X, \mathscr{F}(\mathfrak{M}))$.*

Proof. Suppose F is relatively compact in $(X, weak)$. By Theorem 5, $\sum_{j=1}^{\infty} t_j x_j$ converges in $(X, weak)$ for each $(t_j) \in l^\infty$. Then for every $(s_j) \in l^\infty$, the series $\sum_j s_j x_j$ is subseries convergent in $(X, weak)$. By the Dierolf theorem, $\sum_j s_j x_j$ is subseries convergent in $(X, \mathscr{F}(\mathfrak{M}))$ whenever $(s_j) \in l^\infty$ and, in particular, $\sum_{j=1}^{\infty} s_j x_j$ converges in $(X, \mathscr{F}(\mathfrak{M}))$ for each $(s_j) \in l^\infty$. Then Theorem 5 shows that F is relatively compact in $(X, \mathscr{F}(\mathfrak{M}))$, i.e., (I) holds.

Similarly, Theorem 5 implies (II) and (III). $\square$

Note that $\overline{F} := \overline{F}^\tau$ when we say $\overline{F}$ is (sequentially) compact in (X, τ).

Corollary 4. *Let X be a Hausdorff topological vector space, $\{x_j\} \subset X$ and*

$$F = \left\{ \sum_{j \in \Delta} t_j x_j \colon \Delta \text{ is finite}, \ \sup_j |t_j| \leqslant 1 \right\}.$$

Then we have the following:

(A) *$\overline{F}$ is compact in $(X, weak)$ if and only if $\overline{F}$ is sequentially compact in $(X, \mathscr{F}(\mathfrak{M}))$*;
(B) *$\overline{F}$ is sequentially compact in $(X, weak)$ if and only if $\overline{F}$ is compact in $(X, \mathscr{F}(\mathfrak{M}))$.*

Proof. Suppose that $\overline{F}$ is compact in $(X, weak)$. By Theorem 6, $\overline{F}$ is compact in $(X, \mathscr{F}(\mathfrak{M}))$ and so $\overline{F}$ is sequentially compact in $(X, \mathscr{F}(\mathfrak{M}))$ by Theorem 5. Conversely, if $\overline{F}$ is sequentially compact in $(X, \mathscr{F}(\mathfrak{M}))$, then $\overline{F}$ is compact in $(X, \mathscr{F}(\mathfrak{M}))$ by Theorem 5 and so $\overline{F}$ is compact in $(X, weak)$ by Theorem 6.

Similarly, (B) holds. □

Using Theorem 4 instead of Theorem 5, we have

Theorem 7. *Let X be a Hausdorff topological vector space, $\{x_j\} \subset X$ and $F = \{\sum_{j \in \Delta} x_j \colon \Delta$ is finite$\}$. Then* (I), (II) *and* (III) *in Theorem* 6 *hold for the new defined F. In addition,* (A) *and* (B) *in Corollary* 4 *also hold for the new defined F.*

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